INTERACTION BETWEEN NATURAL AND FORCED CONVECTIONS AT A SPHERE IN A LAMINAR FLOW REGIME

Arnošt KIMLA and Jiří MÍČKA

Department of Mathematics
Prague Institute of Chemical Technology, 160 00 Prague 6

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The authors deal with the formulation and solution of a boundary value problem describing the influence of natural convection on forced convection during diffusion to a sphere in a laminar flow regime, the orientation of the natural convection being opposite to that of the forced one. The results of numerical solution are discussed and compared with the case where the orientation of both types of convection is the same[1]. There is a good qualitative agreement with illustrative physical concepts referring both to hydrodynamics and concentrations.

Natural convection accompanies every diffusion process taking place in the gravitational field. Its influence on the rate of the diffusion transport depends on the geometrical and physical properties of the system in which the diffusion proceeds. The first part of the present work deals with the formulation and numerical solution of the boundary value problem for the natural convection component in a liquid phase during forced laminar flow to a sphere, the direction of the forced convection being opposite to that of the natural one. In the second part, the numerical results obtained are physically interpreted and compared with those corresponding to an analogous model, where the directions of both kinds of streaming are the same[1].

Mathematical Model

The system to be described mathematically consists of a sphere of radius a, placed in a solution with a uniform velocity distribution and with a nearly constant concentration c_0 in a large distance from the sphere, whereas the concentration at the surface of the sphere is equal to zero. The stationary convective diffusion to the flowed-by sphere is then in spherical coordinates r, φ, and ϑ (Fig. 1) described by the following system of partial differential equations for the velocity components v_r, v_ϑ, pressure p, and concentration c (the v_φ component is equal to zero because of symmetry)

$$v_r \partial c / \partial r + r^{-1} v_\vartheta \partial c / \partial \vartheta = D \, \Delta_r c \qquad (1)$$

$$v_r \partial v_r / \partial r + r^{-1} v_\vartheta \partial v_r / \partial \vartheta - r^{-1} v_\vartheta^2 = \varrho^{-1} k g (c - c_0) \cos \vartheta - \\ - \varrho^{-1} \partial p / \partial r + \nu (\Delta_r v_r - 2r^{-2} v_r - 2r^{-2} v_\vartheta \operatorname{cotg} \vartheta - 2r^{-2} \partial v_\vartheta / \partial \vartheta) \qquad (2)$$

$$v_r \partial v_\vartheta/\partial r + r^{-1} v_\vartheta \partial v_\vartheta/\partial \vartheta + r^{-1} v_r v_\vartheta = -\varrho^{-1} k g (c - c_0) \sin \vartheta -$$

$$- (\varrho r)^{-1} \partial p/\partial \vartheta + \nu(\Delta_r v_\vartheta - (r \sin \vartheta)^{-2} v_\vartheta + 2r^{-2} \partial v_r/\partial \vartheta) \qquad (3)$$

$$\partial v_r/\partial r + 2r^{-1} v_r + r^{-1} \partial v_\vartheta/\partial \vartheta + r^{-1} v_\vartheta \operatorname{cotg} \vartheta = 0\,, \qquad (4)$$

where $\Delta r = \partial^2/\partial r^2 + 2r^{-1}\partial/\partial r + r^{-2}\partial^2/\partial \vartheta^2 + r^{-2} \operatorname{cotg} \vartheta \partial/\partial \vartheta$.
The boundary conditions are

$$c(a, \vartheta) = 0\,, \qquad \lim_{r \to \infty} c(r, \vartheta) = c_0 \neq 0\,,$$

$$\partial c/\partial \vartheta(r, 0) = 0, \quad \partial c/\partial \vartheta(r, \pi) = 0\,, \qquad (5)$$

$$v_r(a, \vartheta) = 0, \qquad \lim_{r \to \infty} v_r(r, \vartheta) = v \cos \vartheta\,,$$

$$\partial v_r/\partial \vartheta(r, 0) = 0, \quad \partial v_r/\partial \vartheta(r, \pi) = 0\,, \qquad (6)$$

$$v_\vartheta(a, \vartheta) = 0, \quad \lim_{r \to \infty} v_\vartheta(r, \vartheta) = v \sin \vartheta\,,$$

$$v_\vartheta(r, 0) = 0, \quad v_\vartheta(r, \pi) = 0\,. \qquad (7)$$

Here, D denotes diffusion coefficient, ϱ density of the solution, ν kinematic viscosity, $k = (\partial \varrho/\partial c)_{c = c_0}$, and v velocity of forced streaming.

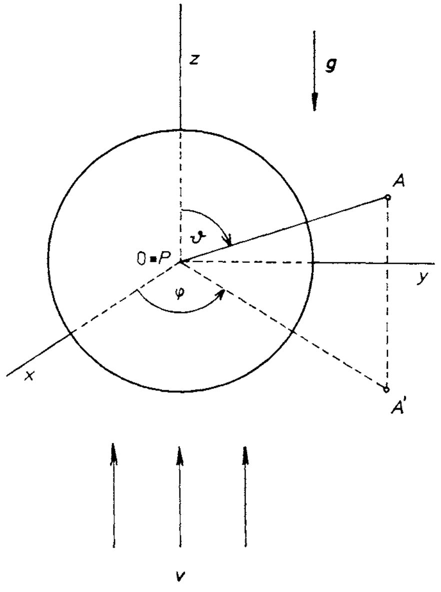

FIG. 1
Orientation of the coordinate system

The boundary value problem can conveniently be simplified[1] by introducing the stream function ψ defined as

$$v_r = r^{-2} \sin^{-1} \vartheta \,.\, \partial\psi/\partial\vartheta, \quad v_\vartheta = -r^{-1} \sin^{-1} \vartheta \,.\, \partial\psi/\partial r\,, \tag{8}$$

eliminating the pressure p, neglecting the quadratic terms of the stream function ψ and introducing the dimensionless variables z, C, Φ and criteria Pe (Peclet number), Gr (Grashof number) and Ra (Rayleigh number) defined as

$$z = 1 - r^{-1}a, \quad C = c_0^{-1}c, \quad \Phi = a(\mathrm{Gr}\, \nu r^2)^{-1}\, \psi\,, \tag{9}$$

$$\mathrm{Pe} = D^{-1}av, \quad \mathrm{Gr} = (\varrho\nu^2)^{-1}\, kgc_0a^3, \quad \mathrm{Ra} = D^{-1}\, \mathrm{Gr}\, \nu\,. \tag{10}$$

The components Φ and C corresponding to natural convection are then given by the following system of partial differential equations[1]:

$$\begin{gathered}(1-z)^3 \sin \vartheta \partial^2 C/\partial z^2 + (1-z) \sin \vartheta \partial^2 C/\partial \vartheta^2 + \\ + \left[2\, \mathrm{Pe} \sin \vartheta \cos \vartheta (\tfrac{1}{2}(1-z) - \tfrac{3}{4}(1-z)^2 + \tfrac{1}{4}(1-z)^4) + \right. \\ \left. + \mathrm{Ra}(1-z)\, \partial\Phi/\partial\vartheta\right] \partial C/\partial z + \left[(1-z) \cos \vartheta - \mathrm{Pe} \sin^2 \vartheta \,. \right. \\ \left. .\, (1 - \tfrac{3}{4}(1-z) - \tfrac{1}{4}(1-z)^3) - \mathrm{Ra}(2\Phi + (1-z)\, \partial\Phi/\partial z)\right] \partial C/\partial\vartheta + \\ + \mathrm{Ra}(1-z)\, \partial\Phi/\partial\vartheta \,.\, \partial C_1/\partial z - \mathrm{Ra}\, \partial C_1/\partial\vartheta(2\Phi + (1-z)\, \partial\Phi/\partial z) = 0\,,\end{gathered} \tag{11}$$

$$\begin{gathered}(1-z)^6\, \partial^4\Phi/\partial z^4 + (1-z)^2\, \partial^4\Phi/\partial\vartheta^4 + 2(1-z)^4\, \partial^4\Phi/\partial z^2\partial\vartheta^2 - \\ - 4(1-z)^5\, \partial^3\Phi/\partial z^3 - 2 \operatorname{cotg} \vartheta(1-z)^2\, \partial^3\Phi/\partial\vartheta^3 - 2 \operatorname{cotg} \vartheta(1-z)^4 \,. \\ .\, \partial^3\Phi/\partial z^2\partial\vartheta + (1-z)^2\, (1 + 3 \sin^{-2} \vartheta)\, \partial^2\Phi/\partial\vartheta^2 - \operatorname{cotg} \vartheta(1-z)^2 \,. \\ .\, (2 + 3 \sin^{-2} \vartheta)\, \partial\Phi/\partial\vartheta - (1-z) \sin^2 \vartheta \,.\, \partial C/\partial z - \sin \vartheta \cos \vartheta \,.\, \partial C/\partial\vartheta = \\ = (1-z) \sin^2 \vartheta \,.\, \partial C_1/\partial z + \sin \vartheta \cos \vartheta \,.\, \partial C_1/\partial\vartheta\,.\end{gathered} \tag{12}$$

Here, C_1 is the concentration component corresponding to forced convection. This system of equations was solved as in ref.[1] with the boundary conditions

$$\begin{aligned} &C(0, \vartheta) = 0, && C(1, \vartheta) = 0\,, \\ &\partial C/\partial\vartheta(z, 0) = 0, && \partial C/\partial\vartheta(z, \pi) = 0\,, \end{aligned} \tag{13}$$

$$\begin{aligned} &\Phi(0, \vartheta) = 0, && \Phi(1, \vartheta) = 0\,, \\ &\Phi(z, 0) = 0, && \Phi(z, \pi) = 0\,, \end{aligned} \tag{14}$$

$$\begin{aligned} &\partial\Phi/\partial z(0, \vartheta) = 0, && \partial\Phi/\partial z(1, \vartheta) = 0\,, \\ &\partial\Phi/\partial\vartheta(z, 0) = 0, && \partial\Phi/\partial\vartheta(z, \pi) = 0\,. \end{aligned} \tag{15}$$

RESULTS AND DISCUSSION

Velocity Field in the Vicinity of the Flowed-by Sphere

The values of the function Φ obtained by numerical solution of the boundary value problem (*11*)–(*15*) and of an analogous problem solved earlier[1] characterize the influence of natural convection on the hydrodynamics in the vicinity of the flowed-by sphere. For a quantitative description of this influence, it is convenient to introduce two dimensionless quantities analogous to the Reynolds number:

$$(\mathrm{Re})_r = av_r/\nu\,, \tag{16}$$

$$(\mathrm{Re})_\vartheta = av_\vartheta/\nu\,. \tag{17}$$

The former quantity characterizes the streaming of the solution in the vicinity of the flowed-by sphere in the radial direction, the latter in the tangential direction. They both depend on the coordinates r, ϑ, eventually z, ϑ, and can be expressed by using the function Φ:

$$(\mathrm{Re})_r = (\mathrm{Gr}/\sin\vartheta)\,\partial\Phi/\partial\vartheta\,, \tag{18}$$

$$(\mathrm{Re})_\vartheta = (\mathrm{Gr}/\sin\vartheta)\,(2\Phi + (1 - z)\,\partial\Phi/\partial z)\,. \tag{19}$$

It follows from the form of Eqs (*16*)–(*19*) and from Eqs (*11*) and (*12*) that both the radial and tangential velocities are directly proportional to Gr at constant Ra and Pe. This is physically understandable, since $\mathrm{Gr} = \mathrm{Ra}\, D/\nu$, i.e. the Grashof number is directly proportional to the diffusion coefficient D and inversely proportional to the viscosity ν at constant Ra. The value of Gr can thus increase either if D increases or if ν decreases. In the former case, the diffusion flux also increases and so do the concentration difference and the force causing natural convection. In the latter case, the hindrance due to viscosity diminishes and the streaming is enhanced. Both these effects may add, since they act in the same sense.

It should be noted that the more exact equations (*1*)–(*7*) involve also quadratic terms of the velocity components, whereby the linearity between the quantities $(\mathrm{Re})_r$ and $(\mathrm{Re})_\vartheta$ may be violated under extreme conditions. However, it can be shown by test programs[2] that the influence of the quadratic terms is negligible in a rather broad region of Gr and Ra values.

Changes in the velocity field depending on Ra at constant Gr can be discussed analogously with respect to the equality $\mathrm{Ra} = \mathrm{Gr}\,\nu/D$. Now, however, the dependence of the velocities on Ra is not linear, since this criterion occurs besides Pe in Eq. (*11*) and thus (through the function C) also in Eq. (*12*), namely not homogeneously in the first power. Moreover, the symmetry is necessarily distorted owing to the mutual orientation of the forced convection and of the field of gravity. If the mutual orientation changes, it brings about a change of the force of buoyancy, which is controlled by the concentration field in the vicinity of the flowed-by sphere. This is illustrated in Figs 2 and 3 showing the quantity

$$V = [(\mathrm{Re})_r^2 + (\mathrm{Re})_\vartheta^2]^{1/2} = a\nu^{-1}(v_r^2 + v_\vartheta^2)^{1/2} \qquad (20)$$

as function of the relative distance $y = (r - a)/a$ for several values of Ra at constant Gr and Pe.

We also followed the dependence of the quantity V on the Peclet number Pe at constant Ra. It can be expected that the fraction of the velocity corresponding to natural convection will decrease with increasing Pe not only relatively but also absolutely. Namely if Pe increases, the thickness of the diffusion layer diminishes[3], this being the region of marked concentration changes. During streaming due to gravity, therefore, there is a stronger hindering effect of the sphere surface on the one hand and of the liquid layers with quasiconstant concentration on the other hand. These effects are illustrated quantitatively in Figs 4 and 5.

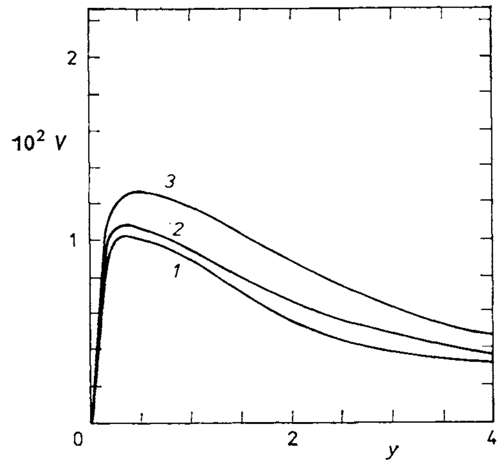

FIG. 2

Dependence of the function V on the distance y (co-current case, $\vartheta = 90°$, Gr = 1, Pe = = 512). Ra: *1* 16; *2* 81; *3* 256

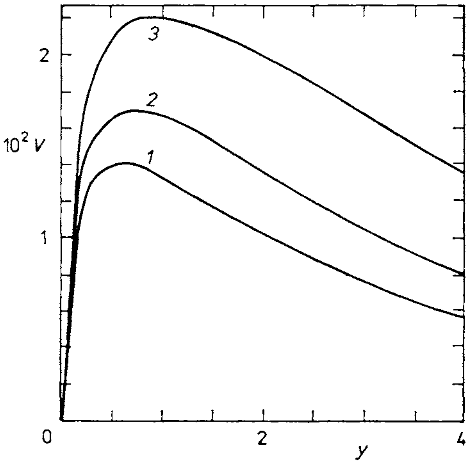

FIG. 3

Dependence of the function V on the distance y (counter-current case, $\vartheta = 90°$, Gr = 1, Pe = 512). Ra: *1* 16; *2* 81; *3* 256

Convective Diffusion

Another function calculated in the present work is C, which has the physical meaning of the difference between the analytical concentration during simultaneous forced and natural convection and the concentration at $Gr = 0$, hence $Ra = 0$, corresponding to "pure" forced convection. Therefore, C acquires negative values in some regions. Knowledge of this function permits to calculate the contribution of the diffusion flux corresponding to natural convection.

The diffusion flux, q, is given by the familiar formula

$$q = c_0 D \operatorname{grad} c(0, \vartheta)$$

and the integral diffusion flux, Q, to the sphere surface can be expressed as

$$Q = 4\pi a^2 c_0 D I ,$$

where

$$I = \frac{1}{2} \int_0^{\pi} \operatorname{grad} C(0, \vartheta) \sin \vartheta d\vartheta . \tag{21}$$

Using numerically calculated approximate values, C_{ij}, of the function C at the grid

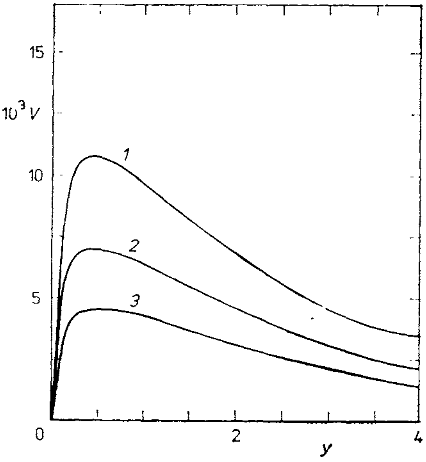

FIG. 4

Dependence of the function V on the distance y (co-current case, $\vartheta = 90°$, $Gr = 1$, $Ra = 81$). Pe: *1* 512; *2* 1000; *3* 1 728

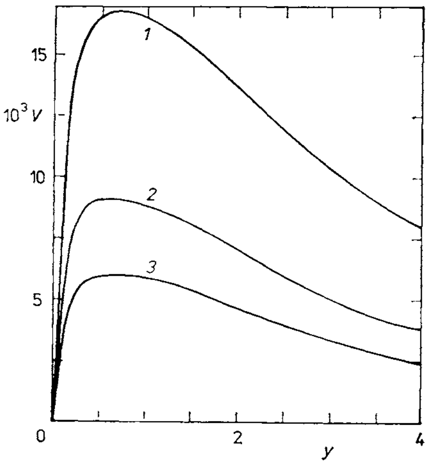

FIG. 5

Dependence of the function V on the distance y (counter-current case, $\vartheta = 90°$, $Gr = 1$, $Ra = 81$). Pe: *1* 512; *2* 1 000; *3* 1 728 points, we calculated the approximate values of the concentration gradient as

$$(\text{grad } C)_j = \tfrac{1}{6}h^{-1}(-11C_{0j} + 18C_{1j} - 9C_{2j} + 2C_{3j}) + O(h^3)\,, \qquad (22)$$

where h denotes the step of the grid used. The small term $O(h^3)$ was neglected. The values thus found are further denoted as $(\text{grad } C)_\text{n}$.

TABLE I

Values of $(\text{grad } C)_\text{n}$ calculated by numerical solution of the boundary value problem

ϑ deg	Ra	$(\text{grad } C)_\text{n}$		
		Pe = 512	Pe = 1 000	Pe = 1 728
0	16	0·01830	0·01086	0·00670
	81	0·10076	0·05760	0·03560
	256	0·39566	0·20436	0·13281
	625	1·38292	0·62384	0·33487
30	16	0·00727	0·00339	0·00159
	81	0·04083	0·01811	0·00859
	256	0·17145	0·06597	0·03577
	625	0·69713	0·21926	0·09824
60	16	−0·00480	−0·00382	−0·00299
	81	−0·02544	−0·02019	−0·01576
	256	−0·08508	−0·07041	−0·05366
	625	−0·24654	−0·19765	−0·12651
90	16	−0·00884	−0·00562	−0·00375
	81	−0·04794	−0·02978	−0·01981
	256	−0·17736	−0·10529	−0·06962
	625	−0·51422	−0·31308	−0·16945
120	16	−0·00888	−0·00517	−0·00329
	81	−0·04832	−0·02740	−0·01739
	256	−0·18181	−0·09702	−0·06136
	625	−0·56588	−0·29190	−0·15055
150	16	−0·00764	−0·00404	−0·00241
	81	−0·04157	−0·02136	−0·01270
	256	−0·15684	−0·07540	−0·04437
	625	−0·50175	−0·22708	−0·10894
180	16	−0·00632	−0·00285	−0·00154
	81	−0·03433	−0·01502	−0·00804
	256	−0·12929	−0·05272	−0·02731
	625	−0·42020	−0·15813	−0·06677

The values of (grad $C)_n$ for selected values of ϑ, Pe, and Ra are summarized in Table I. The results agree with qualitative physical concepts. At the "front" side of the forced streaming (near $\vartheta = 180°$, Fig. 1), the component of the concentration gradient corresponding to natural convection is negative, since the analytical concentration at the sphere surface decreases by natural convection. With decreasing ϑ, the influence of the natural streaming first increases until ϑ reaches about 90°, then decreases, and for low values of ϑ the gradient of C becomes positive. In this region, the natural convection is oriented toward the sphere surface and thus the resulting concentration gradient increases. A comparison of Table I with an analogous table in ref.[1] suggests that also the character of the dependence of grad C on ϑ changes somewhat when the two types of streaming are oriented opposite to each other. Most of the numerical data are higher in absolute values than those given in ref.[1]. This is caused by the character of the concentration field near the sphere during forced convection[3] and the resulting character of the velocity field. The latter was described and discussed in the preceding section.

As in the co-current case[1], the following asymptotical formula can be derived for Pe → ∞ for the counter-current case:

$$\text{grad}\, C(0, \pi) = \text{Ra}\, \text{Pe}^{-1}(k_0 + k_1\, \text{Pe}^{-1/3} + k_2\, \text{Ra}\, \text{Pe}^{-4/3} + \ldots)$$

and, with regard to this, the empirical formula for $\vartheta \in \langle 0, \pi \rangle$

$$\text{grad}\, C(0, \vartheta) = \text{Ra}\, \text{Pe}^{-1}(k_0(\vartheta) + k_1(\vartheta)\, \text{Pe}^{-1/3} + k_2(\vartheta)\, \text{Ra}\, \text{Pe}^{-4/3})\,. \qquad (23)$$

If we introduce this into Eq. (21) we obtain an empirical formula for the auxiliary quantity I:

$$I = \text{Ra}\, \text{Pe}^{-1}(K_0 + K_1\, \text{Pe}^{-1/3} + K_2\, \text{Ra}\, \text{Pe}^{-4/3})\,. \qquad (24)$$

The coefficients in Eq. (23) were determined by using the values of (grad $C)_n$ as described in ref.[1]; thus we were able to calculate the approximate values of grad C from Eq. (23) denoted further as (grad $C)_e$. The results are summarized in Table II together with the deviations in per cent from (grad $C)_n$ calculated as

$$p = \frac{(\text{grad}\, C)_e - (\text{grad}\, C)_n}{(\text{grad}\, C)_n}\,.\, 100\%\,.$$

Thus, it can be seen that the empirical formula (23) gives a good approximation of grad $C(0, \vartheta)$ except for low values of ϑ.

The coefficients in Eq. (24) were calculated analogously as those in Eq. (23) by using numerically calculated values of I (denoted further as I_n). We obtained

$$K_0 = -0{\cdot}4519, \quad K_1 = 2{\cdot}289\,.$$

TABLE II

Values of the coefficients in Eq. (*23*), (grad $C)_e$, and per cent deviations p

ϑ deg	Ra	k_0	k_1	Pe = 512			Pe = 1 000			Pe = 1 728		
				k_2	(grad $C)_e$	p	k_2	(grad $C)_e$	p	k_2	(grad $C)_e$	p
0	16	1·0009	−3·289	3·463	0·0189	3·1	5·136	0·0109	0·2	8·679	0·0068	1·3
	81				0·1041	3·4		0·0578	0·4		0·0357	0·2
	256				0·4031	1·9		0·2057	0·7		0·1236	−7·0
	625				1·3650	−1·3		0·6206	−0·5		0·3575	6·8
30	16	0·0508	1·507	2·026	0·0072	−0·6	2·334	0·0033	−3·2	3·775	0·0017	4·3
	81				0·0442	8·2		0·0179	−1·4		0·0090	4·3
	256				0·1829	6·7		0·0669	1·4		0·0330	−7·6
	625				0·6693	−4·0		0·2171	−1·0		0·1050	6·8
60	16	−0·6616	4·119	−0·365	−0·0046	−3·4	−1·041	−0·0040	5·3	−1·769	−0·0030	−1·0
	81				−0·0244	−4·3		−0·0209	3·6		−0·0152	3·2
	256				−0·0848	−0·4		−0·0707	0·5		−0·0504	−6·1
	625				−0·2471	0·2		−0·1967	−0·5		−0·1344	6·3
90	16	−0·6490	2·947	−0·998	−0·0089	0·6	−2·310	−0·0057	2·0	−3·094	−0·0038	0·2
	81				−0·0475	−0·9		−0·0302	1·5		−0·0195	−1·7
	256				−0·1715	−3·3		−0·1058	0·5		−0·0654	−6·0
	625				−0·5285	2·8		−0·3117	−0·4		−0·1796	6·0
120	16	−0·4985	1·729	−1·220	−0·0088	−0·6	−2·211	−0·0053	1·9	−2·860	−0·0033	0·3
	81				−0·0485	0·4		−0·0278	1·5		−0·0171	−1·4
	256				−0·1793	−1·4		−0·0978	0·8		−0·0577	−5·9
	625				−0·5719	1·1		−0·2899	−0·7		−0·1594	5·9
150	16	−0·2925	0·390	−1·103	−0·0078	1·5	−1·713	−0·0041	1·6	−1·901	−0·0024	0·4
	81				−0·0420	1·1		−0·0217	1·4		−0·0125	−1·3
	256				−0·1563	−0·3		−0·0761	0·9		−0·0420	−5·4
	625				−0·5030	0·3		−0·2254	−0·8		−0·1147	5·3
180	16	−0·0934	−0·863	−0·932	−0·0064	1·3	−1·313	−0·0029	2·2	−0·905	−0·0015	0·0
	81				−0·0348	1·2		−0·0154	2·6		−0·0079	−1·6
	256				−0·1297	0·4		−0·0546	5·5		−0·0261	−4·3
	625				−0·4192	−0·2		−0·1636	3·5		−0·0696	4·3

TABLE III
Values of I from Eq. (*24*)

Pe	Ra	K_2	I_n	I_e	q
512	16	−0·388	−0·00530	−0·00523	−1·28
	81		−0·02844	−0·02744	−3·50
	256		−0·10134	−0·09501	−6·24
	625		−0·25928	−0·27463	5·92
1 000	16	−1·205	−0·00348	−0·00360	3·54
	81		−0·01840	−0·01885	2·45
	256		−0·06460	−0·06499	0·59
	625		−0·18743	−0·18645	−0·52
1 728	16	−1·518	−0·00244	−0·00243	−0·49
	81		−0·01285	−0·01252	−2·61
	256		−0·04397	−0·04147	−5·70
	625		−0·10492	−0·11100	5·80

The values of K_2, I_n, and I from Eq. (*24*) (denoted as I_e) are summarized in Table III together with the per cent deviations

$$q = \frac{I_e - I_n}{I_n} \,.\, 100\% \,.$$

In analysing the possible errors in grad C, we found very good agreement of the errors with those found in the numerical solution of the analogous co-current case[1]. Therefore, we do not mention them here.

It can be concluded that the influence of natural convection on forced convection in the counter-current case is, except for the orientation of the natural convective diffusion component, not appreciably different from that in the co-current case. It can be expected that in the general case where the direction of the forced streaming is at an angle $\alpha \in (0, \pi)$ to the direction of the gravity force the results will lie between those for the two extreme cases of $\alpha = 0$ and $\alpha = \pi$.

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